

# TWISTED COVERS AND SPECIALIZATIONS

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**ABSTRACT.** The central topic is this question: is a given  $k$ -étale algebra  $\prod_l E_l/k$  the specialization of a given  $k$ -cover  $f : X \rightarrow B$  at some point  $t_0 \in B(k)$ ? Our main tool is a *twisting lemma* that reduces the problem to finding  $k$ -rational points on a certain  $k$ -variety. Previous forms of this twisting lemma are generalized and unified. New applications are given: a Grunwald form of Hilbert's irreducibility theorem over number fields, a non-Galois variant of the Tchebotarev theorem for function fields over finite fields, some general specialization properties of covers over PAC or ample fields.

## 1. PRESENTATION

**1.1. The central question.** If  $f : X \rightarrow B$  is an algebraic cover defined over a field  $k$  and  $t_0$  a  $k$ -rational point on  $B$ , not in the branch locus of  $f$ , the specialization of  $f$  at  $t_0$  is defined as a finite  $k$ -étale algebra of degree  $n = \deg(f)$ . For example, if  $B = \mathbb{P}^1$  and  $f$  is given by some polynomial  $P(T, Y) \in k[T, Y]$ , it is the product of separable field extensions of  $k$  that correspond to the irreducible factors of  $P(t_0, Y)$  (for all but finitely many  $t_0 \in k$ ). Our central question is whether a given degree  $n$   $k$ -étale algebra  $\prod_l E_l/k$  is the specialization of some given degree  $n$   $k$ -cover  $f : X \rightarrow B$  at some unramified point  $t_0 \in B(k)$ . The classical Hilbert specialization property corresponds to the special case for which étale algebras are taken to be single degree  $n$  field extensions and the answer is positive for at least one of them.

The question has already been investigated in [DG11a] and [DG11b] for regular Galois covers and in [DL11] for covers with geometric monodromy group  $S_n$  (definitions recalled in §2.2). The aim of this paper is to handle the situation of arbitrary covers, to provide a unifying approach and to give further applications.

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**1.2. The twisting lemma.** Our main tool is a *twisting lemma* that gives a general answer to the question: under certain hypotheses, *answer is Yes if there exist unramified  $k$ -rational points on the covering space  $\tilde{X}$  of certain twisted covers  $\tilde{f} : \tilde{X} \rightarrow B$* . This lemma has several variants. The first one, for regular Galois covers, was established in [Dèb99] for covers of  $\mathbb{P}^1$  and in [DG11a] for a general base space. It is used in [DL11] to obtain the second one, for covers with geometric monodromy group  $S_n$ . We will prove the two variants shown on top row of the following diagram, which indicates that they generalize the two previous ones, shown on bottom row.

$$\begin{array}{ccc} \text{Galois} & \Leftrightarrow & \text{general} \\ \Downarrow & & \Downarrow \\ \text{regular Galois} & \Rightarrow & \text{monodromy } S_n \end{array}$$

The *Galois* variant is for the situation  $f : X \rightarrow B$  is a Galois cover, regular or not; it is proved in §3.1. The *general* variant is proved in §3.2 and concerns arbitrary covers, Galois or not, regular or not. Implication  $\Rightarrow$  in upper row means that the general variant will be obtained from the Galois variant. We will also be interested in the converse in the twisting lemma: answer to the original question is Yes *if and only if* there exist unramified  $k$ -rational points on the twisted covers.

The twisting lemma is a geometric *avatar* of an argument of Tchebotarev known as the *Field Crossing Argument* and which notably appears in the proof of the Tchebotarev density theorems over global fields and in the theory of PAC fields (see [FJ04]). The twisting lemma formalizes the core of the argument and produces a geometric tool: the variety  $\tilde{X}$ . This allows a unifying approach over an arbitrary base field: questions are reduced to finding rational points on  $\tilde{X}$ . Letting the base field vary then yields previous results in various contexts and leads to new applications. The twisted cover  $\tilde{f} : \tilde{X} \rightarrow B$ , which appeared first in [Dèb99], could also be defined by using the language of torsors. Another related approach using an embedding problem presentation has also been recently proposed by Bary-Soroker [BS10].

**1.3. Applications.** As in previous papers, they are obtained over fields with good arithmetic properties: PAC fields, finite fields, number fields, ample fields. We present them below in connection with those from previous works.

**1.3.1. Over a PAC field  $k$**  (definition recalled in §4.1), the regular Galois variant was used in [Dèb99] to prove that, given a group  $G$  and a

subgroup  $H \subset G$ , any Galois extension  $E/k$  of group  $H$  is a specialization of any regular Galois  $k$ -cover  $f : X \rightarrow \mathbb{P}^1$  of group  $G$  (thereby proving the so-called Beckmann-Black conjecture for PAC fields). A not necessarily Galois analog with an arbitrary degree  $n$   $k$ -étale algebra  $\prod_l E_l/k$  replacing  $E/k$  is proved in [DL11] under the assumption that  $f$  is a degree  $n$   $k$ -cover of geometric monodromy group  $S_n$ . Corollary 4.1 is a refinement of the first result (the regularity assumption is relaxed) while corollary 4.2 is a variant of the second one (allowing more general monodromy groups). Similar applications have been obtained by Bary-Soroker [BS10].

The general spirit of these results is that over a PAC field there is no diophantine obstruction<sup>1</sup> to a given étale algebra being a specialization of some given cover; obstructions only come from Galois theory. This has some impact on the arithmetic of PAC fields. For example a by-product of [DL11] is that if  $k$  is a PAC field of characteristic 0 (for simplicity), every degree  $n$  extension  $E/k$  can be realized by some trinomial  $Y^n - Y + b$  with  $b \in k$ .

1.3.2. *Over a finite field  $k = \mathbb{F}_q$* , the twisting lemma can be combined with Lang-Weil to obtain an estimate for the number of points  $t_0 \in \mathbb{F}_q$  at which a given degree  $n$  étale algebra  $\prod_l E_l/\mathbb{F}_q$  is a specialization of a given degree  $n$   $\mathbb{F}_q$ -cover  $f : X \rightarrow \mathbb{P}^1$  of geometric monodromy group  $S_n$  (corollary 4.3). This type of result is known in the literature as *Tchebotarev theorem for function fields over finite fields*. For example, if  $\prod_l E_l/\mathbb{F}_q$  is the single degree  $n$  field extension  $\mathbb{F}_{q^n}/\mathbb{F}_q$ , the estimate is of the form  $q/n + O(\sqrt{q})$ . In the specific case where  $f$  is given by the trinomial  $Y^n + Y - T$ , it yields results of Cohen and Ree proving a conjecture of Chowla. See §4.2 for details and references.

For finite fields  $\mathbb{F}_q$ , the same general spirit as for PAC fields can be retained — no diophantine obstruction to the problem —, but provided that  $q$  be suitably large.

1.3.3. *The local-global situation* of a number field  $k$  given with some completions  $k_v$  was central in [DG11a]. The main result was a Hilbert-Grunwald theorem showing that every regular Galois  $k$ -cover  $f : X \rightarrow \mathbb{P}^1$  of group  $G$  has specializations at points  $t_0 \in k$  that are *Galois field extensions of group  $G$*  (Hilbert) with the extra property (Grunwald) that *they induce prescribed unramified extensions  $E^v/k_v$  of Galois group  $H_v \subset G$  at each finite place  $v$  in a given finite set  $S$* , the only condition on the places being that the residue fields be suitably big and of order

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<sup>1</sup>in the sense that existence of rational points on some variety, which is a condition of our twisting lemma in general, is automatic over a PAC field  $k$ .

prime to  $|G|$ . An analog is given in [DL11] for not necessarily Galois covers: the Hilbert condition becomes that the specialization at  $t_0$  is *a degree  $n$  field extension* and the Grunwald condition that *the local degrees* are imposed at each  $v \in S$ ; this is proved under the assumption that  $f$  is a degree  $n$   $k$ -cover of geometric monodromy group  $S_n$ .

§4.3 has a similar local-global flavor. The outcome is a generalization to general regular covers  $f : X \rightarrow \mathbb{P}^1$  of the non-Galois analog above (corollary 4.5). On the way the following typical result of Fried is reproved (and generalized): if the Galois group  $\overline{G} \subset S_n$  over  $\overline{\mathbb{Q}}(T)$  of a degree  $n$  polynomial  $P(T, Y) \in \mathbb{Q}(T)[Y]$  contains a  $n$ -cycle, then the associated Hilbert subset contains infinitely many arithmetic progressions with ratio a prime number. See §4.3 for details and references.

Here it is the relative flexibility of the local extensions obtained from global specializations that is the striking phenomenon. In the Galois situation, the very existence of global extensions with such local properties may sometimes even be questioned. Recall for example that results from [DG11a] lead to some obstruction to the Regular Inverse Galois Problem (yet unproved to be not vacuous) related to some analytic questions around the Tchebotarev density theorem.

Other local-global situations can be considered, for example that of a base field that is a function field  $\kappa(x)$  with  $\kappa$  either a suitably large finite field or a PAC field with enough cyclic extensions. We refer to [DG11b] where these situations have been considered.

1.3.4. *Over ample fields* (definition recalled in §4.4), the twisting lemma leads to this general property of ample fields (corollary 4.6): if a  $k$ -cover  $f : X \rightarrow B$  of curves specializes to some  $k$ -étale algebra  $\prod_l E_l/k$  at some unramified point  $t_0 \in B(k)$ , then it specializes to the same  $k$ -étale algebra  $\prod_l E_l/k$  at infinitely many points  $t \in B(k)$ .

## 2. BASICS

Given a field  $k$ , we fix an algebraic closure  $\overline{k}$  and denote the separable closure of  $k$  in  $\overline{k}$  by  $k^{\text{sep}}$  and its absolute Galois group by  $G_k$ . If  $k'$  is an overfield of  $k$ , we use the notation  $\otimes_k k'$  for the scalar extension from  $k$  to  $k'$ : for example, if  $X$  is a  $k$ -curve,  $X \otimes_k k'$  is the  $k'$ -curve obtained by scalar extension. For more on this section, we refer to [DD97, §2] or [Dèb09, chapitre 3].

**2.1. Étale algebras and their Galois representations.** Given a field  $k$ , a  *$k$ -étale algebra* is a product  $\prod_{l=1}^s E_l/k$  of  $k$ -isomorphism classes of finite sub-field extensions  $E_1/k, \dots, E_s/k$  of  $k^{\text{sep}}/k$ . Set  $m_l = [E_l : k]$ ,  $l = 1, \dots, s$  and  $m = \sum_{l=1}^s m_l$ . If  $N/k$  is a Galois

extension containing the Galois closures of  $E_1/k, \dots, E_s/k$ , the Galois group  $\text{Gal}(N/k)$  acts by left multiplication on the left cosets of  $\text{Gal}(N/k)$  modulo  $\text{Gal}(N/E_l)$  for each  $l = 1, \dots, s$ . The resulting action  $\text{Gal}(N/k) \rightarrow S_m$  on all these left cosets, which is well-defined up to conjugation by elements of  $S_m$ , is called the *Galois representation of  $\prod_{l=1}^s E_l/k$  relative to  $N$* . Equivalently it can be defined as the action of  $\text{Gal}(N/k)$  on the set of all  $k$ -embeddings  $E_l \hookrightarrow N$ ,  $l = 1, \dots, s$ .

Conversely, an action  $\mu : \text{Gal}(N/k) \rightarrow S_m$  determines a  $k$ -étale algebra in the following way. For  $i = 1, \dots, m$ , denote the fixed field in  $N$  of the subgroup of  $\text{Gal}(N/k)$  consisting of all  $\tau$  such that  $\mu(\tau)(i) = i$  by  $E_i$ . The product  $\prod_l E_l/k$  for  $l$  ranging over a set of representatives of the orbits of the action  $\mu$  and where each extension  $E_l/k$  is regarded modulo  $k$ -isomorphism is a  $k$ -étale algebra with  $\sum_l [E_l : k] = m$ .

*G-Galois variant:* if  $\prod_{l=1}^s E_l/k$  is a *single Galois extension*  $E/k$ , the restriction  $\text{Gal}(N/k) \rightarrow \text{Gal}(E/k)$  is called *the G-Galois representation of  $E/k$  (relative to  $N$ )*. Any map  $\varphi : \text{Gal}(N/k) \rightarrow G$  obtained by composing  $\text{Gal}(N/k) \rightarrow \text{Gal}(E/k)$  with a monomorphism  $\text{Gal}(E/k) \rightarrow G$  is called a *G-Galois representation of  $E/k$  (relative to  $N$ )*. The extension  $E/k$  can be recovered from  $\varphi : \text{Gal}(N/k) \rightarrow G$  by taking the fixed field in  $N$  of  $\ker(\varphi)$ . One obtains the Galois representation  $\text{Gal}(N/k) \rightarrow S_n$  of  $E/k$  (relative to  $N$ ) from a G-Galois representation  $\varphi : \text{Gal}(N/k) \rightarrow G$  (relative to  $N$ ) by composing it with the left-regular representation of the image group  $\varphi(\text{Gal}(N/k))$ .

**2.2. Covers and function field extensions.** Given a regular projective geometrically irreducible  $k$ -variety  $B$ , a  *$k$ -cover of  $B$*  is a finite and generically unramified morphism  $f : X \rightarrow B$  defined over  $k$  with  $X$  a normal and irreducible variety. Through the function field functor  $k$ -covers  $f : X \rightarrow B$  correspond to finite separable field extensions  $k(X)/k(B)$ . The  $k$ -cover  $f : X \rightarrow B$  is said to be *Galois* if the field extension  $k(X)/k(B)$  is; if in addition  $f : X \rightarrow B$  is given together with an isomorphism  $G \rightarrow \text{Gal}(k(X)/k(B))$ , it is called a  *$k$ -G-Galois cover of group  $G$* .

A  $k$ -cover  $f : X \rightarrow B$  is said to be *regular* if  $k(X)$  is a regular extension of  $k$ , *i.e.* if  $k(X) \cap \bar{k} = k$ , or equivalently, if  $X$  is geometrically irreducible. In general, there is some *constant extension* in  $f : X \rightarrow B$ , which we denote by  $\widehat{k}_f/k$  and is defined by  $\widehat{k}_f = k(X) \cap k^{\text{sep}}$  (the special case  $\widehat{k}_f = k$  corresponds to the situation  $f : X \rightarrow B$  is regular).

If  $f : X \rightarrow B$  is a  $k$ -cover, its Galois closure over  $k$  is a Galois  $k$ -cover  $g : Z \rightarrow B$ , which *via* the cover-field extension dictionary, corresponds to the Galois closure of  $k(X)/k(B)$ . The Galois group

$\text{Gal}(k(Z)/k(B))$  is called the *monodromy group* of  $f$ . Denote next by  $k^{\text{sep}}(Z)$  the *compositum* of  $k(Z)$  and  $k^{\text{sep}}$  (in a fixed separable closure of  $k(B)$ )<sup>2</sup>. The Galois group  $\text{Gal}(k^{\text{sep}}(Z)/k^{\text{sep}}(B))$  is called the *geometric monodromy group* of  $f$ ; it is a normal subgroup of the monodromy group  $\text{Gal}(k(Z)/k(B))$ . The *branch divisor* of the  $k$ -cover  $f$  is the formal sum of all hypersurfaces of  $B \otimes_k k^{\text{sep}}$  such that the associated discrete valuations are ramified in the field extension  $k^{\text{sep}}(Z)/k^{\text{sep}}(B)$ .

If  $f : X \rightarrow B$  is regular,  $f \otimes_k k^{\text{sep}}$  is a  $k^{\text{sep}}$ -cover, the Galois closure of its function field extension is  $k^{\text{sep}}(Z)/k^{\text{sep}}(B)$  and its branch divisor is the same as the branch divisor of  $f$ , and it is the formal sum of all hypersurfaces of  $B \otimes_k k^{\text{sep}}$  such that the associated discrete valuations are ramified in the field extension  $k^{\text{sep}}(X)/k^{\text{sep}}(B)$ .

**2.3.  $\pi_1$ -representations.** Given a reduced positive divisor  $D \subset B$ , denote the  $k$ -fundamental group of  $B \setminus D$  by  $\pi_1(B \setminus D, t)_k$  where  $t \in B(\bar{k}) \setminus D$  is a base point (which corresponds to the choice of an algebraic closure of  $k(B)$ ). Conjoining the two dictionaries covers-function field extensions and field extensions-Galois representations, we obtain the following correspondences:  $k$ -covers of  $B$  of degree  $n$  (resp.  $k$ -G-Galois covers of  $B$  of group  $G$ ) with branch divisor contained in  $D$  correspond to transitive morphisms  $\phi : \pi_1(B \setminus D, t)_k \rightarrow S_n$  (resp. to epimorphisms  $\phi : \pi_1(B \setminus D, t)_k \rightarrow G$ ). The regularity property corresponds to the extra condition that the restriction of  $\phi$  to  $\pi_1(B \setminus D, t)_{k^{\text{sep}}}$  remains transitive (resp. remains onto). These morphisms are called *fundamental group representations* ( $\pi_1$ -representations for short) of the corresponding  $k$ -covers and  $k$ -G-Galois covers.

**2.4. Specializations.** Each  $k$ -rational point  $t_0 \in B(k) \setminus D$  provides a section  $s_{t_0} : G_k \rightarrow \pi_1(B \setminus D, t)_k$  to the exact sequence

$$1 \rightarrow \pi_1(B \setminus D, t)_{k^{\text{sep}}} \rightarrow \pi_1(B \setminus D, t)_k \rightarrow G_k \rightarrow 1$$

well-defined up to conjugation by elements in  $\pi_1(B \setminus D, t)_{k^{\text{sep}}}$ .

If  $\phi : \pi_1(B \setminus D, t)_k \rightarrow G$  represents a  $k$ -G-Galois cover  $f : X \rightarrow B$ , the morphism  $\phi \circ s_{t_0} : G_k \rightarrow G$  is a G-Galois representation. The fixed field in  $k^{\text{sep}}$  of  $\ker(\phi \circ s_{t_0})$  is the residue field at some/any point above  $t_0$  in the extension  $k(X)/k(B)$ . We denote it by  $k(X)_{t_0}$  and call  $k(X)_{t_0}/k$  the *specialization* of the  $k$ -G-Galois cover  $f$  at  $t_0$ .

If  $\phi : \pi_1(B \setminus D, t)_k \rightarrow S_n$  represents a  $k$ -cover  $f : X \rightarrow B$ , the morphism  $\phi \circ s_{t_0} : G_k \rightarrow S_n$  is the *specialization representation* of  $f$  at  $t_0$ . The corresponding  $k$ -étale algebra is denoted by  $\prod_{l=1}^s k(X)_{t_0, l}/k$  and

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<sup>2</sup>Note that as  $g : Z \rightarrow B$  is Galois,  $k(Z)$  only depends on the  $k(B)$ -isomorphism class of  $k(X)/k(B)$  (but not on  $k(X)/k(B)$  itself).

called the *collection of specializations* of  $f$  at  $t_0$ . Each field  $k(X)_{t_0,l}$  is a residue extension at some prime above  $t_0$  in the extension  $k(X)/k(B)$  and *vice-versa*;  $k(X)_{t_0,l}$  is called a *specialization* of  $f$  at  $t_0$ . The *compositum* in  $k^{\text{sep}}$  of the Galois closures of all specializations at  $t_0$  is the specialization at  $t_0$  of the Galois closure of  $f$  (viewed as a  $k$ -G-Galois cover). If the  $k$ -cover  $f$  is regular, the fields  $k(X)_{t_0,l}$  correspond to the definition fields of the points in the fiber  $f^{-1}(t_0)$  and  $\phi \circ \mathbf{s}_{t_0} : G_k \rightarrow S_n$  to the *action* of  $G_k$  on them.

### 3. THE TWISTING LEMMA

Given a field  $k$ , the question we address is whether a given  $k$ -cover specializes to some given  $k$ -étale algebra at some unramified  $k$ -rational point. We first consider the situation of Galois covers in §3.1 and then handle the non-Galois situation in §3.2 by “going to the Galois closure”. The Galois situation was considered in [DG11a] in the special case of *regular* Galois covers. But the Galois closure of a  $k$ -cover is not regular in general, even if  $f : X \rightarrow B$  is regular, and this special case needs to be extended. §3.1 is a generalization of the twisting lemma from [DG11a] to not necessarily regular Galois covers.

**3.1. The twisting lemma for Galois covers.** Fix the field  $k$  and a Galois  $k$ -cover  $g : Z \rightarrow B$ . Denote its branch divisor by  $D$ , the Galois group  $\text{Gal}(k(Z)/k(B))$  by  $G$ , the  $\pi_1$ -representation of the  $k$ -G-Galois cover  $g : Z \rightarrow B$  by  $\phi : \pi_1(B \setminus D, t)_k \rightarrow G$ , the geometric monodromy group  $\text{Gal}(k^{\text{sep}}(Z)/k^{\text{sep}}(B))$  by  $\overline{G}$  and the constant extension in  $g : Z \rightarrow B$  by  $\widehat{k}_g/k$ .

**3.1.1. Twisting Galois covers.** Let  $N/k$  be some Galois extension with Galois group  $H$  isomorphic to a subgroup of  $G$ . With no loss we may and will view  $H$  itself as a subgroup of  $G$ . We assume the following *compatibility condition* of  $N/k$  with the constant extension  $\widehat{k}_g/k$ :

(const/comp) *the fixed field  $N^{H \cap \overline{G}}$  of  $H \cap \overline{G}$  in  $N$  is the field  $\widehat{k}_g$ .*

This condition is trivially satisfied in the regular case as both fields  $N^{H \cap \overline{G}}$  and  $\widehat{k}_g$  equal  $k$ .

Consider the homomorphism  $\Lambda : G_k \rightarrow G/\overline{G}$  induced by  $\phi$  on the quotient  $G_k = \pi_1(B \setminus D, t)_k / \pi_1(B \setminus D, t)_{k^{\text{sep}}}$ . The map  $\Lambda$  is a  $G$ -Galois representation of the constant extension  $\widehat{k}_g/k$  (relative to  $k^{\text{sep}}$ ); it is called the *constant extension map* [DD97, §2.8]. As it is surjective, we have  $\text{Gal}(\widehat{k}_g/k) \simeq G/\overline{G}$  and so condition (const/comp) implies that  $H\overline{G} = G$ .

Let  $\varphi : G_k \rightarrow H$  be the  $G$ -Galois representation of the Galois extension  $N/k$  (relative to  $k^{\text{sep}}$ ) and  $\overline{\varphi} : G_k \rightarrow G/\overline{G}$  be the composed map of  $\varphi$  with the canonical surjection  $\overline{\cdot} : G \rightarrow G/\overline{G}$ . Hypothesis (const/comp) rewrites as follows:

(const/comp) *There exists  $\overline{\chi} \in \text{Aut}(G/\overline{G})$  such that  $\Lambda = \overline{\chi} \circ \overline{\varphi}$ .*

(The equivalence follows from  $\widehat{k}_g = (k^{\text{sep}})^{\ker(\Lambda)}$  and

$$(k^{\text{sep}})^{\ker(\overline{\varphi})} = ((k^{\text{sep}})^{\ker(\varphi)})^{\ker(\overline{\varphi})/\ker(\varphi)} = N^{\varphi(\ker(\overline{\varphi}))} = N^{H \cap \overline{G}}.$$

Also note that as  $\Lambda : G_k \rightarrow G/\overline{G}$  is onto, an automorphism  $\overline{\chi}$  satisfying (const/comp) is unique).

Assume there exists an isomorphism  $\chi : H \rightarrow H'$  onto a subgroup  $H' \subset G$  that induces  $\overline{\chi}$  modulo  $\overline{G}$ . With  $\text{Per}(G)$  the permutation group of  $G$ , consider then the map

$$\widetilde{\phi}^{\chi\varphi} : \pi_1(B \setminus D, t)_k \rightarrow \text{Per}(G)$$

defined by this formula, where  $r$  is the restriction  $\pi_1(B \setminus D, t)_k \rightarrow G_k$ : for  $\theta \in \pi_1(B \setminus D, t)_k$  and  $x \in G$ ,

$$\widetilde{\phi}^{\chi\varphi}(\theta)(x) = \phi(\theta) x (\chi \circ \varphi \circ r)(\theta)^{-1}$$

It is easily checked that  $\widetilde{\phi}^{\chi\varphi}$  is a group homomorphism. However the corresponding action of  $\pi_1(B \setminus D, t)_k$  on  $G$  is not transitive in general. More precisely we have the following.

**Lemma 3.1.** *Under hypothesis (const/comp), we have  $\widetilde{\phi}^{\chi\varphi}(\theta)(\overline{G}) \subset \overline{G}$  for every  $\theta \in \pi_1(B \setminus D, t)_k$ .*

*Proof.* For all  $\theta \in \pi_1(B \setminus D, t)_k$  and  $x \in \overline{G}$ , we have:

$$\overline{\widetilde{\phi}^{\chi\varphi}(\theta)(x)} = \overline{\phi(\theta) \cdot x \cdot (\chi \circ \varphi \circ r)(\theta)^{-1}} = \Lambda(r(\theta)) \cdot \overline{\chi}(\varphi(r(\theta)))^{-1} = 1 \quad \square$$

Consider the morphism, denoted by  $\widetilde{\phi}_G^{\chi\varphi} : \pi_1(B \setminus D, t)_k \rightarrow \text{Per}(\overline{G})$ , that sends  $\theta \in \pi_1(B \setminus D, t)_k$  to the restriction of  $\widetilde{\phi}^{\chi\varphi}(\theta)$  on  $\overline{G}$ . Its restriction  $\pi_1(B \setminus D, t)_{k^{\text{sep}}} \rightarrow \text{Per}(\overline{G})$  is given by

$$\widetilde{\phi}_G^{\chi\varphi}(\theta)(x) = \phi(\theta) x \quad (\theta \in \pi_1(B \setminus D, t)_{k^{\text{sep}}}, x \in \overline{G})$$

Thus this restriction is obtained by composing the original  $\pi_1$ -representation  $\phi$  restricted to  $\pi_1(B \setminus D, t)_{k^{\text{sep}}}$  with the left-regular representation  $\overline{G} \rightarrow \text{Per}(\overline{G})$  of  $\overline{G}$ . This shows that  $\widetilde{\phi}_G^{\chi\varphi} : \pi_1(B \setminus D, t)_k \rightarrow \text{Per}(\overline{G})$  is the  $\pi_1$ -representation of some regular  $k$ -cover, which we denote by  $\widetilde{g}^{\chi\varphi} : \widetilde{Z}^{\chi\varphi} \rightarrow B$  and call the *twisted cover* of  $g$  by  $\chi\varphi$ .



3.1.2. *Statement of the twisting lemma for Galois covers.* The following statement gives the main property of the twisted cover.

Some notation is needed. Conjugation automorphisms in some group  $\mathcal{G}$  are denoted by  $\text{conj}(\omega)$  for  $\omega \in \mathcal{G}$ :  $\text{conj}(\omega)(x) = \omega x \omega^{-1}$  ( $x \in \mathcal{G}$ ). The set of all isomorphisms  $\chi : H \rightarrow H'$  onto a subgroup  $H' \subset G$  that induce  $\bar{\chi}$  modulo  $\bar{G}$  is denoted by  $\text{Isom}_{\bar{\chi}}(H, H')$ .

Fix then a set  $\{\chi_\gamma : H \rightarrow H_\gamma \mid \gamma \in \Gamma\}$  of representatives of all isomorphisms  $\chi \in \text{Isom}_{\bar{\chi}}(H, H')$  with  $H'$  ranging over all subgroups of  $G$  isomorphic to  $H$ , modulo the equivalence that identifies  $\chi_1 \in \text{Isom}_{\bar{\chi}}(H, H'_1)$  and  $\chi_2 \in \text{Isom}_{\bar{\chi}}(H, H'_2)$  if  $H'_2 = \omega H'_1 \omega^{-1}$  and  $\chi_2 \chi_1^{-1} = \text{conj}(\omega)$  for some  $\omega \in \bar{G}$ .

**Twisting lemma 3.2** (Galois form). *Under condition (const/comp), we have the following conclusions (a) and (b).*

(a) *For each subgroup  $H' \subset G$  isomorphic to  $H$ , each  $\chi \in \text{Isom}_{\bar{\chi}}(H, H')$  and each  $t_0 \in B(k) \setminus D$ , these conditions are equivalent:*

- (i) *there exists a point  $x_0 \in \tilde{Z}^{\chi\varphi}(k)$  such that  $\tilde{g}^{\chi\varphi}(x_0) = t_0$ ,*
- (ii) *there is  $\omega \in \bar{G}$  such that  $(\phi \circ \mathbf{s}_{t_0})(\tau) = \omega(\chi \circ \varphi)(\tau)\omega^{-1}$ ,  $\tau \in G_k$ , (where  $\mathbf{s}_{t_0} : G_k \rightarrow \pi_1(B \setminus D, t)_k$  is the section associated with  $t_0$ ).*

(b) *For each  $t_0 \in B(k) \setminus D$ , the following are equivalent:*

- (iii) *the specialization  $k(Z)_{t_0}/k$  of the  $k$ - $G$ -Galois cover  $g : Z \rightarrow B$  is the extension  $N/k$ ,*
- (iv) *there exists an isomorphism  $\chi \in \text{Isom}_{\bar{\chi}}(H, \phi \circ \mathbf{s}_{t_0}(G_k))$  such that conditions (i)-(ii) hold for this  $\chi$ ,*
- (v) *there exists  $\gamma \in \Gamma$  such that conditions (i)-(ii) hold for  $\chi = \chi_\gamma$ .*

*Furthermore an element  $\gamma \in \Gamma$  as in (v) is necessarily unique.*

A single twisted cover is involved in (a) while there are several in (b). In this respect the representation viewpoint used in (a) may look more natural than the field extension one in (b). The latter however is more useful in practice. Also note that conditions (iv)-(v), being equivalent to (iii), do not depend on the chosen  $\pi_1$ -representation  $\phi : \pi_1(B \setminus D, t)_k \rightarrow G$  of  $g : Z \rightarrow B$  modulo conjugation by elements of  $G$ .

*Remark 3.3.* (a) Existence of some subgroup  $H' \subset G$  such that the set  $\text{Isom}_{\bar{\chi}}(H, H')$  is non-empty, which amounts to  $\Gamma \neq \emptyset$ , is not guaranteed; in this case conditions (iii)-(iv)-(v) fail. It is however guaranteed under each of the assumptions  $\bar{\chi} = \text{Id}_{G/\bar{G}}$  or  $\text{Out}(G/\bar{G}) = \{1\}$ . Indeed if  $\bar{\chi} = \text{Id}_{G/\bar{G}}$ , then  $\text{Id}_H \in \text{Isom}_{\bar{\chi}}(H, H)$ , and if  $\text{Out}(G/\bar{G}) = \{1\}$ , i.e., if every automorphism of  $G/\bar{G}$  is inner, then, as  $H\bar{G} = G$ , every inner automorphism  $\text{conj}(\bar{\omega})$  with  $\bar{\omega} \in G/\bar{G}$  lifts to some isomorphism

$\text{conj}(\omega) : H \rightarrow H$  with  $\omega \in H$ . Both assumptions includes the regular case as then  $G/\overline{G} = \{1\}$ .

(b) Some uniqueness property can be added to (iv), as in (v). Indeed an isomorphism  $\chi \in \text{Isom}_{\overline{\chi}}(H, \phi \circ \mathbf{s}_{t_0}(G_k))$  satisfying conditions (i)-(ii), as the one in (iv), is necessarily unique up to left composition by  $\text{conj}(\omega)$  with  $\omega \in \text{Nor}_{\overline{G}}(\phi \circ \mathbf{s}_{t_0}(G_k))$ . The advantage of condition (v) is that the set  $\bigcup_{\gamma \in \Gamma} \tilde{Z}^{\chi_\gamma \varphi}(k)$  where unramified  $k$ -rational points should be found to conclude that (iii) holds does not depend on  $t_0$  (although the element  $\gamma \in \Gamma$  in (v) does). Moreover the uniqueness property in (v) makes it easier to count the points  $t_0 \in B(k)$  for which (iii) holds.

(c) The proof of (i)  $\Leftrightarrow$  (ii) below shows further that the number of  $k$ -rational points on  $\tilde{Z}^{\chi \varphi}$  above some given unramified point  $t_0 \in B(k)$ , if positive, is equal to the order of the group  $\text{Cen}_{\overline{G}}(\chi(H))$ .

**3.1.3. Proof of the twisting lemma 3.2.** (a) Fix a subgroup  $H' \subset G$  isomorphic to  $H$ , an isomorphism  $\chi \in \text{Isom}_{\overline{\chi}}(H, H')$  and a point  $t_0 \in B(k) \setminus D$ . The map  $\tilde{\phi}_G^{\chi \varphi} \circ \mathbf{s}_{t_0} : G_k \rightarrow \text{Per}(\overline{G})$  is the action of  $G_k$  on the fiber  $(\tilde{g}^{\chi \varphi})^{-1}(t_0)$ ; it is given by

$$\tilde{\phi}_G^{\chi \varphi}(\mathbf{s}_{t_0}(\tau)) = \phi(\mathbf{s}_{t_0}(\tau)) x (\chi \circ \varphi)(\tau)^{-1} \quad (\tau \in G_k, x \in \overline{G})$$

Elements  $\tilde{\phi}_G^{\chi \varphi}(\mathbf{s}_{t_0}(\tau))$  have a common fixed point  $\omega \in \overline{G}$  if and only if  $\phi(\mathbf{s}_{t_0}(\tau)) = \omega (\chi \circ \varphi)(\tau) \omega^{-1}$  ( $\tau \in G_k$ ). This yields (i)  $\Leftrightarrow$  (ii). Furthermore, the set of all  $\omega \in \overline{G}$  satisfying the preceding condition, if non empty, is a left coset  $\omega_0 \text{Cen}_{\overline{G}}(\chi(H))$ ; this proves remark 3.3 (c).

(b) Fix  $t_0 \in B(k) \setminus D$  and a representative of the section  $\mathbf{s}_{t_0} : G_k \rightarrow \pi_1(B \setminus D, t)_k$  (defined up to conjugation in  $\pi_1(B \setminus D, t)_{k^{\text{sep}}}$ ).

Implication (iv)  $\Rightarrow$  (iii) follows from the fact that if  $\chi \in \text{Isom}_{\overline{\chi}}(H, \phi \circ \mathbf{s}_{t_0}(G_k))$  satisfies (i)-(ii), then  $\ker(\phi \circ \mathbf{s}_{t_0})$  and  $\ker(\varphi)$  are equal, and so so are their fixed fields in  $k^{\text{sep}}$ . Conversely assume that the extensions  $k(Z)_{t_0}/k$  and  $N/k$  are equal, *i.e.*  $\ker(\phi \circ \mathbf{s}_{t_0})$  and  $\ker(\varphi)$  are the same subgroup, say  $\mathcal{K}$ , of  $G_k$ . The two morphisms  $\phi \circ \mathbf{s}_{t_0} : G_k \rightarrow \phi \circ \mathbf{s}_{t_0}(G_k) \subset G$  and  $\varphi : G_k \rightarrow H \subset G$  then differ from  $G_k \rightarrow G_k/\mathcal{K}$  by some isomorphisms  $\phi \circ \mathbf{s}_{t_0}(G_k) \rightarrow G_k/\mathcal{K}$  and  $H \rightarrow G_k/\mathcal{K}$ , respectively. Thus they differ from one another by an isomorphism  $\chi : H \rightarrow \phi \circ \mathbf{s}_{t_0}(G_k)$ :  $\phi \circ \mathbf{s}_{t_0} = \chi \circ \varphi$ . It follows from this and from uniqueness of  $\overline{\chi}$  satisfying (const/comp) that  $\chi$  automatically induces  $\overline{\chi}$  modulo  $\overline{G}$ . Conclude that  $\chi \in \text{Isom}_{\overline{\chi}}(H, \phi \circ \mathbf{s}_{t_0}(G_k))$  and conditions (i)-(ii) hold for this  $\chi$ .

Assume (v) holds, *i.e.*, for some  $\gamma \in \Gamma$ , condition (i)-(ii) are satisfied for the isomorphism  $\chi_\gamma : H \rightarrow H_\gamma$  and some  $\omega \in \overline{G}$ . It readily

follows that  $\chi = \text{conj}(\omega) \circ \chi_\gamma$  also satisfies (ii) and is in  $\text{Isom}_{\overline{\chi}}(H, \phi \circ \mathbf{s}_{t_0}(G_k))$ . This establishes (iv). Conversely assume (iv) holds. Let  $\chi \in \text{Isom}_{\overline{\chi}}(H, \phi \circ \mathbf{s}_{t_0}(G_k))$  be an isomorphism such that conditions (i)-(ii) hold, for some  $\omega \in \overline{G}$ . There exist  $\gamma \in \Gamma$  and  $\omega' \in \overline{G}$  such that  $\chi = \text{conj}(\omega') \circ \chi_\gamma$ . It follows that condition (ii) holds for  $\chi_\gamma$  as well (with conjugation factor  $\omega\omega'$ ). Uniqueness of  $\gamma \in \Gamma$  in condition (v) readily follows from condition (ii) and the definition of the set  $\{\chi_\gamma \mid \gamma \in \Gamma\}$ .  $\square$

**3.2. The general form of the twisting lemma.** We fix a degree  $n$   $k$ -cover  $f : X \rightarrow B$  and a degree  $n$   $k$ -étale algebra  $\prod_{l=1}^s E_l/k$  and the question we address is whether  $\prod_{l=1}^s E_l/k$  is the collection  $\prod_l k(X)_{t_0,l}/k$  of specializations of  $f : X \rightarrow B$  at some point  $t_0 \in B(k)$ .

**3.2.1. Statement of the result.** Denote the branch divisor of  $f : X \rightarrow B$  by  $D$ , its Galois closure by  $g : Z \rightarrow B$ , the Galois group  $\text{Gal}(k(Z)/k(B))$  by  $G$ , the  $\pi_1$ -representation of the  $k$ -G-Galois cover  $g : Z \rightarrow B$  by  $\phi : \pi_1(B \setminus D, t)_k \rightarrow G$ , the Galois representation of the field extension  $k(X)/k(B)$  relative to  $k(Z)$  by  $\nu : G \rightarrow S_n$ , the geometric monodromy group  $\text{Gal}(k^{\text{sep}}(Z)/k^{\text{sep}}(B))$  by  $\overline{G}$  and the constant extension in  $g : Z \rightarrow B$  by  $\widehat{k}_g/k$ .

Let  $N/k$  be the *compositum* inside  $k^{\text{sep}}$  of the Galois closures of the extensions  $E_l/k$ ,  $l = 1, \dots, s$ , and  $H = \text{Gal}(N/k)$ . A necessary condition for a positive answer to the question is that  $N$  be the *compositum* inside  $k^{\text{sep}}$  of the Galois closures of the extensions  $k(X)_{t_0,l}/k$ . In particular,  $H$  should be isomorphic to some subgroup of  $G$ . From now on we will assume it. With no loss we may then and will view  $H$  as a subgroup of  $G$ . Finally let  $\varphi : G_k \rightarrow H$  be the  $G$ -Galois representation of  $N/k$  relative to  $k^{\text{sep}}$  and  $\mu : H \rightarrow S_n$  be the Galois representation of  $\prod_{l=1}^s E_l/k$  relative to  $N$ .

Some further notation from §3.1 is retained. The constant extension compatibility condition (const/comp) determines a unique automorphism  $\overline{\chi}$  of  $G/\overline{G}$  (§3.1.1). The twisted cover  $\widetilde{g}^{\chi\varphi} : \widetilde{Z}^{\chi\varphi} \rightarrow B$  is defined for every isomorphism  $\chi : H \rightarrow H'$  onto a subgroup  $H' \subset G$  inducing  $\overline{\chi}$  modulo  $\overline{G}$  (§3.1.1). The set of all such isomorphisms  $\chi : H \rightarrow H'$  is denoted by  $\text{Isom}_{\overline{\chi}}(H, H')$ . The isomorphisms  $\chi_\gamma : H \rightarrow H_\gamma$  ( $\gamma \in \Gamma$ ) are defined in §3.1.2.

**Twisting lemma 3.4** (general form). *Let  $f : X \rightarrow B$  be a  $k$ -cover and  $\prod_{l=1}^s E_l/k$  be a  $k$ -étale algebra as above. Assume further that condition (const/comp) from §3.1.1 holds for the Galois closure  $g : Z \rightarrow B$  of  $f$ . Then for each  $t_0 \in B(k) \setminus D$ , the following conditions are equivalent:*

- (i)  $\prod_l E_l/k$  is the collection  $\prod_l k(X)_{t_0,l}/k$  of specializations of  $f : X \rightarrow B$  at the point  $t_0$ .

(ii) *there is a subgroup  $H' \subset G$  isomorphic to  $H$  and an isomorphism  $\chi \in \text{Isom}_{\overline{\chi}}(H, H')$  such that*

1. *there exists  $x_0 \in \widetilde{Z}^{\chi\varphi}(k)$  with  $\widetilde{g}^{\chi\varphi}(x_0) = t_0$ , and*
2. *there exists  $\sigma \in S_n$  that  $\nu \circ \chi(h) = \sigma \mu(h) \sigma^{-1}$  for every  $h \in H$ .*

*Furthermore if (ii) holds, it holds for some isomorphism  $\chi_\gamma : H \rightarrow H_\gamma$  for some  $\gamma \in \Gamma$  and the element  $\gamma$  is then necessarily unique.*

3.2.2. *About condition (ii-2).* We focus on condition (ii-2) which is the group-theoretical part of condition (ii) (while condition (ii-1) is the diophantine part).

We first note for later use that the number of  $\gamma \in \Gamma$  for which condition (ii-2) holds for  $\chi = \chi_\gamma$ , if positive, is equal to the number of distinct isomorphisms  $\chi_\gamma, \chi_{\gamma'}$  ( $\gamma, \gamma' \in \Gamma$ ) such that the actions  $\nu \circ \chi_\gamma : H \rightarrow S_n$  and  $\nu \circ \chi_{\gamma'} : H \rightarrow S_n$  are conjugate in  $S_n$ .

Below we give three standard situations where condition (ii-2) holds.

- (a) *geometric monodromy group  $S_n$ :*  $G = \overline{G} = S_n$  as in [DL11]. Condition (const/comp) holds and  $\nu : S_n \rightarrow S_n$  is the natural action:  $\nu = \text{Id}_{S_n}$ . Condition  $\nu \circ \chi_\gamma(h) = \sigma \mu(h) \sigma^{-1}$  ( $h \in H$ ) is satisfied with  $\chi_\gamma$  the representative of the isomorphism  $\mu : H \rightarrow S_n$  (and some  $\sigma \in S_n$ ).
- (b) *Galois situation:*  $f : X \rightarrow B$  is a Galois  $k$ -cover,  $\prod_l E_l/k$  is a Galois field extension  $E/k$  of group  $H \subset G$  and  $\Gamma \neq \emptyset$ . Then  $\nu$  is the left-regular representation  $G \rightarrow \text{Per}(G)$  and  $\mu$  its restriction  $H \rightarrow \text{Per}(G)$ . Note next that if  $\gamma \in \Gamma$ , the restriction  $\nu|_H : H \rightarrow \text{Per}(G)$  and  $\nu \circ \chi_\gamma : H \rightarrow \text{Per}(G)$  are conjugate actions. Condition (ii-2) follows.

In (c) below, the *type of a permutation*  $\sigma \in S_n$  is the (multiplicative) divisor of all lengths of disjoint cycles involved in the cycle decomposition of  $\sigma$  (for example, an  $n$ -cycle is of type  $n^1$ ).

- (c) *cyclic specializations:* condition (const/comp) holds,  $H$  is a cyclic subgroup of  $G$  generated by an element  $\omega$  such that  $\nu(\omega)$  is of type equal to the divisor of all degrees  $[E_l : k]$  of field extensions in the étale algebra  $\prod_l E_l/k$ .

Indeed for every integer  $a \geq 1$  such that  $(a, |H|) = 1$ , let  $\chi_a : H \rightarrow H$  be the morphism that maps  $\omega$  to  $\omega^a$ . As  $H\overline{G} = G$ , each map  $\chi_a$  induces an automorphism of the cyclic group  $G/\overline{G}$ . Then there is necessarily an integer  $a \geq 1$  such that  $\chi_a$  induces  $\overline{\chi}$  modulo  $\overline{G}$  and  $(a, |H|) = 1$ <sup>3</sup>. From the hypothesis, the types of  $\nu(\omega)$  and  $\mu(\omega)$  are the same. But so

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<sup>3</sup>An exercise: this amounts to showing that if  $b$  is an integer prime to  $\nu = |G/\overline{G}|$  and  $|G| = \mu\nu$ , then there exists an integer  $a = b + k\nu$  that is prime to  $\mu\nu$ . Take for  $k$  the product of the prime divisors of  $\mu$  that do not divide  $b$ .

are the types of  $\nu(\omega)$  and  $\nu \circ \chi_a(\omega)$ . Conclude that the actions  $\nu \circ \chi_a$  and  $\mu$  are conjugate.

**3.2.3. Comparizon with previous forms.** We compare the general form (lemma 3.4) with the Galois form (lemma 3.2) and the geometric monodromy group  $S_n$  form [DL11, lemma 2.1] of the twisting lemmas.

*Lemma 3.4 (general form)  $\Rightarrow$  lemma 3.2 (Galois form):* Both forms have the assumption (const/comp). In the Galois situation from §3.1, the  $k$ -cover is *Galois* (and so  $f = g$ ) and the  $k$ -étale algebra is a *Galois* field extension  $E/k$  with group  $\text{Gal}(E/k) = H$  (so  $\prod_{l=1}^s E_l/k = E/k$  and  $N = E$ ). Then statement (i)  $\Leftrightarrow$  (ii) in lemma 3.4 exactly corresponds to statement (iii)  $\Leftrightarrow$  (v) in lemma 3.2.

Indeed condition (ii) from lemma 3.4 reduces to its first part (ii-1) (see §3.2.2 (b)) and then coincides with condition (v) from lemma 3.2, and condition (i) from lemma 3.4 corresponds to condition (iii) from lemma 3.2 (note that the étale algebra  $\prod_l E_l/k$  (resp.  $\prod_l k(X)_{t_0,l}/k$ ) from condition (i) is a product of  $|G|/|H|$  copies of the Galois field extension  $E/k$  (resp.  $k(X)_{t_0}/k$ )).

*Lemma 3.4 (general form)  $\Rightarrow$  lemma 2.1 from [DL11]:* In [DL11], the  $k$ -cover  $f : X \rightarrow B$  is of degree  $n$  and geometric monodromy group  $S_n$ . Then  $G = \overline{G} = S_n$ , that is, we are in the standard situation (a) from §3.2.2. Thus condition (ii-2) holds. The twisted cover  $\tilde{g}^N : \tilde{Z}^N \rightarrow B$  in [DL11, lemma 2.1] is the twisted cover  $\tilde{g}^{\mu\varphi} : \tilde{Z}^N \rightarrow B$  in this paper. Conclude that (i)  $\Rightarrow$  (ii) in [DL11, lemma 2.1] exactly corresponds to (ii)  $\Rightarrow$  (i) in lemma 3.4.

**3.2.4. Proof of the twisting lemma 3.4.** We will use the Galois form of the twisting lemma to establish the general form.

(i)  $\Rightarrow$  (ii): Assume (i) holds. Necessarily  $N$  is the compositum of the Galois closures of the extensions  $k(X)_{t_0,l}/k$ . From the twisting lemma 3.2 for Galois covers, there is a unique  $\gamma \in \Gamma$  satisfying condition (ii-1) from lemma 3.4. And from lemma 3.2 (a), this last condition is equivalent to existence of some  $\omega \in \overline{G}$  such that  $(\phi \circ s_{t_0})(\tau) = \omega (\chi_\gamma \circ \varphi)(\tau) \omega^{-1}$  for all  $\tau \in G_k$ . Thus we obtain:

$$(\nu \circ \phi \circ s_{t_0})(\tau) = \nu(\omega) (\nu \circ \chi_\gamma \circ \varphi)(\tau) \nu(\omega)^{-1} \quad (\tau \in G_k)$$

But condition (i) gives  $\nu \circ \phi \circ s_{t_0}(\tau) = \beta \mu \circ \varphi(\tau) \beta^{-1}$  ( $\tau \in G_k$ ), for some  $\beta \in S_n$ . Conjoining these equalities yields condition (ii-2).

(ii)  $\Rightarrow$  (i): Assume (ii) holds. From lemma 3.2, existence of  $x_0 \in \tilde{Z}^{\chi\varphi}(k)$  such that  $\tilde{g}^{\chi\varphi}(x_0) = t_0$  implies that  $N$  is the compositum of the Galois

closures of the  $k(X)_{t_0, l}$ , and so we have  $(\phi \circ \mathbf{s}_{t_0})(\tau) = \omega(\chi \circ \varphi)(\tau) \omega^{-1}$  for some  $\omega \in \overline{G}$  and all  $\tau \in G_k$ .

Denote the orbits of  $\mu : H \rightarrow S_n$ , which correspond to the extensions  $E_1, \dots, E_s$ , by  $\mathcal{O}_1, \dots, \mathcal{O}_s$ . Fix one of them, *i.e.*  $l \in \{1, \dots, s\}$ , and let  $i \in \{1, \dots, n\}$  be some index such that  $E_l$  is the fixed field in  $k^{\text{sep}}$  of the subgroup of  $G_k$  fixing  $i$  via the action  $\mu \circ \varphi$ . For  $j = \nu(\omega)(\sigma(i))$  (with  $\sigma$  given by condition (ii-2)), we have

$$\begin{aligned} (\nu \circ \phi \circ \mathbf{s}_{t_0})(\tau)(j) &= \nu(\omega) (\nu \circ \chi \circ \varphi)(\tau) (\sigma(i)) \\ &= \nu(\omega) (\text{conj}(\sigma) \circ \mu \circ \varphi)(\tau) (\sigma(i)) \\ &= \nu(\omega) \sigma (\mu \circ \varphi)(\tau)(i) \end{aligned}$$

and so  $j$  is fixed by  $(\nu \circ \phi \circ \mathbf{s}_{t_0})(\tau)$  if and only if  $i$  is fixed by  $(\mu \circ \varphi)(\tau)$ . Conclude that the specialization  $k(X)_{t_0, j}$  is the field  $E_l$ .  $\square$

#### 4. APPLICATIONS

**4.1. PAC fields.** Recall that a field  $k$  is said to be PAC if every non-empty geometrically irreducible  $k$ -variety has a Zariski-dense set of  $k$ -rational points. If  $k$  is PAC, the twisting lemma leads to the following statements in the two standard situations (b) and (c) from §3.2.2 (the standard situation (a) corresponds to corollary 3.1 from [DL11]). Similar applications over PAC fields can also be found in Bary-Soroker's works [BS10] [BS09].

**Corollary 4.1.** *Let  $k$  be a PAC field,  $f : X \rightarrow B$  be a  $k$ -G-Galois cover of group  $G$  and geometric monodromy group  $\overline{G}$ , and let  $E/k$  be a Galois extension of group  $H \subset G$ . Assume that condition (const/comp) from §3.1 holds and  $\text{Out}(G/\overline{G}) = \{1\}$ . Then  $E/k$  is the specialization  $k(X)_{t_0}/k$  of  $f$  at each point  $t_0$  in a Zariski-dense subset of  $B(k) \setminus D$ .*

The special case  $G = \overline{G}$  corresponds to theorem 3.2 of [Dèb99] (which proved the Beckmann-Black conjecture over PAC fields).

*Proof.* Assumption  $\text{Out}(G/\overline{G}) = \{1\}$  assures that  $\Gamma \neq \emptyset$  (remark 3.3 (a)). Pick  $\gamma \in \Gamma$ . Since  $k$  is PAC, the variety  $\tilde{Z}^{\chi_\gamma \varphi}$  has a Zariski-dense set  $\mathcal{Z}$  of  $k$ -rational points. From lemma 3.2, the Zariski-dense subset  $\tilde{Z}^{\chi_\gamma \varphi}(\mathcal{Z}) \setminus D \subset B(k) \setminus D$  satisfies the announced conclusion.  $\square$

**Corollary 4.2.** *Let  $k$  be a PAC field,  $f : X \rightarrow B$  be a degree  $n$   $k$ -cover and let  $1^{\beta_1} \dots n^{\beta_n}$  be the type of some element of the monodromy group  $G$  in the Galois representation  $\nu : G \rightarrow S_n$  of  $k(X)/k(B)$ . Let  $\prod_l E_l/k$  be an étale algebra such that*

- the divisor of all degrees  $[E_l : k]$  is  $1^{\beta_1} \dots n^{\beta_n}$ ,
- condition (const/comp) holds,

- the compositum  $N/k$  of the Galois closures of the extensions  $E_l/k$  is a cyclic extension of order  $\text{ppcm}\{i \mid \beta_i \neq 0\}$ .

Then  $\prod_l E_l/k$  is the collection  $\prod_l k(X)_{t_0, l}/k$  of specializations of  $f$  at each point  $t_0$  in a Zariski-dense subset of  $B(k) \setminus D$ .

A useful special case is for  $1^{\beta_1} \cdots n^{\beta_n} = n^1$ : it can then be concluded that  $f : X \rightarrow B$  specializes to some degree  $n$  field extension at each  $t_0$  in a Zariski-dense subset of  $B(k) \setminus D$  (i.e. the Hilbert irreducibility conclusion) under the assumptions that there is a  $n$ -cycle in  $\nu(G)$  and  $k$  has a degree  $n$  cyclic extension satisfying condition (const/comp). This can be compared to [BS09, corollary 1.4] (and [DL11, corollary 3.1]) which has the same Hilbert conclusion under the assumptions that  $G = \overline{G} = S_n$  and  $k$  has a degree  $n$  separable extension.

*Proof.* Let  $\omega \in G$  with  $\nu(\omega)$  of type  $1^{\beta_1} \cdots n^{\beta_n}$ . Identify the Galois group  $H = \text{Gal}(N/k)$  with the subgroup  $\langle \omega \rangle \subset G$ . We are in the standard situation (c) from §3.2.2 and so condition (ii-2) from lemma 3.4 holds for some isomorphism  $\chi_\gamma$  ( $\gamma \in \Gamma$ ). Since  $k$  is PAC, condition (ii-1) holds for all  $t_0$  in a Zariski-dense subset of  $B(k) \setminus D$ . Therefore condition (i) from lemma 3.4 holds as well, thus ending the proof.  $\square$

**4.2. Finite fields.** If  $k$  is a suitably large finite field  $\mathbb{F}_q$ , the Lang-Weil estimates can be used to guarantee that the twisted covers have  $\mathbb{F}_q$ -rational points. More specifically we have the following result, where we take  $B = \mathbb{P}^1$  for simplicity.

**Corollary 4.3.** *Let  $f : X \rightarrow \mathbb{P}^1$  be a regular  $\mathbb{F}_q$ -cover of degree  $n \geq 2$ , with  $r$  branch points and with geometric monodromy group  $S_n$ . Let  $m_1, \dots, m_s$  be some positive integers (possibly repeated) such that  $\sum_{l=1}^s m_l = n$ . Then the number  $\mathcal{N}(f, m_1, \dots, m_s)$  of unramified points  $t_0 \in \mathbb{F}_q$  such that  $\prod_{l=1}^s \mathbb{F}_{q^{m_l}}/\mathbb{F}_q$  is the collection of specializations of  $f$  at  $t_0$  can be evaluated as follows:*

$$\left| \mathcal{N}(f, m_1, \dots, m_s) - \frac{(q+1) |m_1^1 \cdots m_s^1|}{n!} \right| \leq rn! \sqrt{q}$$

where  $|m_1^1 \cdots m_s^1|$  is the number of elements in the conjugacy class in  $S_n$  corresponding to the type  $m_1^1 \cdots m_s^1$ .

This extends similar estimates that have appeared in the literature for Galois covers under the name of Tchebotarev theorem for function fields over finite fields. See [Wei48], [Fri74] [Eke90], [FJ04, §6], and also [DG11b, corollary 3.5] where the Galois analog of corollary 4.3 is obtained as the outcome of our approach in the standard situation §3.2.2 (b).

For the type  $m_1^1 \cdots m_s^1 = n^1$  of  $n$ -cycles, we obtain that the number  $\mathcal{N}(f, n)$  is asymptotic to  $q/n$  when  $q \rightarrow +\infty$ . For example if  $f : X \rightarrow \mathbb{P}^1$  over  $\mathbb{F}_p$  is given by the trinomial  $Y^n + Y - T$  (which satisfies the assumptions of corollary 4.3 if  $p \nmid n(n-1)$  [Ser92, §4.4]), the number of irreducible trinomials  $Y^n + Y + a \in \mathbb{F}_p[Y]$  realizing the extension  $\mathbb{F}_{p^n}/\mathbb{F}_p$  is asymptotic to  $p/n$  as  $p \rightarrow \infty$ , a result due to Cohen [Coh70] and Ree [Ree71] proving a conjecture of Chowla [Cho66].

*Proof.* We are in the standard situation  $G = \overline{G} = S_n$ . Condition (const/comp) trivially holds. Furthermore, it follows from the beginning note of §3.2.2 that the number of  $\gamma \in \Gamma$  for which condition (ii-2) holds is 1; denote by  $\chi_0$  the corresponding isomorphism. From lemma 3.4, the set of unramified  $\mathbb{F}_q$ -rational points on the twisted variety  $\tilde{Z}^{\chi_0\varphi}$  maps *via* the cover  $\tilde{g}^{\chi_0\varphi} : \tilde{Z}^{\chi_0\varphi} \rightarrow \mathbb{P}^1$  to the set of points  $t_0 \in \mathbb{P}^1(\mathbb{F}_q)$  satisfying the desired conclusion. Using remark 3.3 (c), we obtain

$$0 \leq \frac{|\tilde{Z}^{\chi_0\varphi}(\mathbb{F}_q)|}{|\text{Cen}_{S_n}(\chi_0(H))|} - \mathcal{N}(f, m_1, \dots, m_s) \leq \frac{r(n! - 1)}{|\text{Cen}_{S_n}(\chi_0(H))|}$$

where  $H = \text{Gal}(\mathbb{F}_{q^M}/\mathbb{F}_q)$  with  $M = \text{ppcm}(m_1, \dots, m_s)$  and the term  $(r(n! - 1))$  is an upper bound for the number of ramified points on  $\tilde{Z}^{\chi_0\varphi}$ . Also note that  $\tilde{g}^{\chi_0\varphi}$  and  $g$  being isomorphic over  $k^{\text{sep}}$ , they have the same branch point number, which is the branch point number  $r$  of  $f$ , and that the curves  $\tilde{Z}^{\chi_0\varphi}$  and  $Z$  have the same genus, say  $g$ .

The cyclic subgroup  $\chi_0(H) \subset S_n$  is generated by a permutation of type  $m_1^1 \cdots m_s^1$  (condition (ii-2) from lemma 3.4). Hence we have  $|\text{Cen}_{S_n}(\chi_0(H))| = n! / |m_1^1 \cdots m_s^1|$ . The Lang-Weil estimates give:

$$||\tilde{Z}^{\chi_0\varphi}(\mathbb{F}_q)| - (q + 1)| \leq 2g\sqrt{q}$$

The Riemann-Hurwitz formula yields  $g \leq (r - 2)(n - 1)/2$ . The announced estimate easily follows. (We use that the largest cardinality of a conjugacy class in  $S_n$  is  $n(n - 2)!$ , *i.e.*, that of the class of  $n - 1$ -cycles; this enables us to write that  $(n - 1)/|\text{Cen}_{S_n}(\chi_0(H))| \leq 1$ ).  $\square$

**4.3. Number fields.** Over number fields, we will follow a local-global approach as in [DL11] and [DG11a]. We start with a local result at one prime. We give two versions: a *mere version* for a cover  $f : X \rightarrow \mathbb{P}^1$  and a *G-version* for a G-Galois cover  $g : Z \rightarrow \mathbb{P}^1$ .

For the next two statements, let  $k$  be a number field,  $f : X \rightarrow \mathbb{P}^1$  be a degree  $n$  regular  $k$ -cover,  $r$  be the branch point number,  $G$  (resp.  $\overline{G}$ ) be the monodromy group (resp. geometric monodromy group),  $g : Z \rightarrow \mathbb{P}^1$  be the Galois closure of  $f$ ,  $\nu : G \rightarrow S_n$  be the Galois representation



of  $k(X)/k(T)$  (relative to  $k(Z)$ ) and  $\widehat{k}_g/k$  be the constant extension in  $g$ . A prime number  $p$  is said to be *bad* if it is one from the finite list of primes for which the branch divisor is not étale or there is vertical ramification at  $p$  [DG11a], it is said to be *good* otherwise.

**Corollary 4.4.** *Suppose given*

(in the mere version): *the type  $1^{\beta_1} \dots n^{\beta_n}$  of an element of  $\nu(\overline{G}) \subset S_n$ ,*

(in the G-version): *an element  $\omega \in \overline{G}$ .*

*Then for each prime  $p \geq 4r^2(n!)^2$ , good and totally split in  $\widehat{k}_g/\mathbb{Q}$ , there exists an integer  $b_p \in \mathbb{Z}$  such that for each integer  $t_0 \equiv b_p \pmod{p}$ ,*

(mere version) *the collection of specializations of  $f \otimes_k \mathbb{Q}_p$  at  $t_0$  is an étale algebra  $\prod_l E_l/\mathbb{Q}_p$  with degree divisor  $\prod_l [E_l : \mathbb{Q}_p]^1 = 1^{\beta_1} \dots n^{\beta_n}$ ,*

(G-version) *the specialization of the  $\mathbb{Q}_p$ -G-Galois cover  $g \otimes_k \mathbb{Q}_p$  at  $t_0$  is the unramified extension  $N_p/\mathbb{Q}_p$  of degree  $|\langle \omega \rangle|$ .*

The mere version generalizes theorem 4 from [Fri74]: if  $\nu(\overline{G})$  contains an  $n$ -cycle, then, for  $1^{\beta_1} \dots n^{\beta_n} = n^1$ , the conclusion, stated as in [Fri74] in the situation  $f$  is given by a polynomial  $P(T, Y)$ , is that  $P(t_0, Y)$  is irreducible in  $\mathbb{Q}_p[Y]$ , and so in  $k[Y]$  too.

*Proof.* Consider first the mere version. Let  $p$  be a totally split prime in the extension  $\widehat{k}_g/\mathbb{Q}$  (infinitely many such primes exist from the Tchebotarev density theorem). In particular  $\mathbb{Q}_p \widehat{k}_g = \mathbb{Q}_p$ . For each  $i = 1, \dots, n$  with  $\beta_i > 0$ , let  $E^{p,i}/\mathbb{Q}_p$  be the unique unramified extension of  $\mathbb{Q}_p$  of degree  $i$ . Here we use the twisting lemma 3.4 in the “cyclic specializations” standard situation (c) from §3.2.2; we apply it to the cover  $f \otimes_k \mathbb{Q}_p$  and the  $\mathbb{Q}_p$ -étale algebra  $\prod_i (E^{p,i}/\mathbb{Q}_p)^{\beta_i}$ , where the exponent  $\beta_i$  indicates that the extension  $E^{p,i}/\mathbb{Q}_p$  appears  $\beta_i$  times. Condition (const/comp) holds by definition of  $\widehat{k}_g$  and condition (ii-2) from lemma 3.4 holds for some isomorphism  $\chi_\gamma$ ,  $\gamma \in \Gamma$  (§3.2.2 (c)). If  $p$  is a good prime, the twisted curve  $\widetilde{Z}^{\chi_\gamma \varphi} \otimes_k \mathbb{Q}_p$  has good reduction, and the Lang-Weil estimates then show that if  $p \geq 4r^2(n!)^2$ , the special fiber has at least one unramified  $\mathbb{F}_p$ -rational point; see [DG11a] for more details. From Hensel’s lemma, such a  $\mathbb{F}_p$ -rational point lifts to a  $\mathbb{Q}_p$ -rational point on  $\widetilde{Z}^{\chi_\gamma \varphi}$ . Conclude with lemma 3.4 that the étale algebra  $\prod_i (E^{p,i}/\mathbb{Q}_p)^{\beta_i}$  is the collection of specializations of  $f \otimes_k \mathbb{Q}_p$  at each point  $t_0$  in a coset of  $\mathbb{Z}_p$  modulo  $p\mathbb{Z}_p$ .

The G-version is very similar, but it is the Galois form of the twisting lemma (lemma 3.2) that should be applied, to the regular  $\mathbb{Q}_p$ -G-Galois cover  $g \otimes_k \mathbb{Q}_p$  and the unramified extension of  $\mathbb{Q}_p$  of degree  $|\langle \omega \rangle|$ .  $\square$

Corollary 4.4 can be used simultaneously for several types of elements in  $\nu(\overline{G}) \subset S_n$  and for several elements of  $\overline{G}$ . The weak approximation property of  $\mathbb{P}^1$  then provides arithmetic progressions  $(am + b)_{m \in \mathbb{Z}} \subset \mathbb{Z}$  with ratio  $a$  the product of several corresponding primes. In particular by using all non-trivial elements of  $\overline{G}$ , it can be guaranteed that the specialization at  $am + b$  (for every  $m \in \mathbb{Z}$ ) of the  $\widehat{k}_g$ -G-Galois cover  $g \otimes_k \widehat{k}_g$  be a Galois extension of group  $\overline{G}$ ; this uses a standard argument (recalled in [DG11a, §3.4]) based on a lemma of Jordan. This implies that the specialization at  $am + b$  of the  $k$ -G-Galois cover  $g$  is a Galois extension of group a subgroup of  $G$  containing  $\overline{G}$ . As the  $k$ -cover  $f : X \rightarrow \mathbb{P}^1$  is assumed to be regular (and so  $\nu(\overline{G})$  is a transitive subgroup of  $S_n$ ), it follows that the collection of specializations at  $am + b$  of the  $k$ -cover  $f$  is a single field extension of degree  $n$ , *i.e.* Hilbert's conclusion holds at  $am + b$  (for every  $m \in \mathbb{Z}$ ).

We obtain the following statement, which generalizes [DL11, corollary 4.1] to arbitrary regular covers.

The constants however are not as good as in the “ $G = \overline{G} = S_n$ ” situation of [DL11] because of the preliminary condition on the primes  $p$  that uses the Tchebotarev theorem.

**Corollary 4.5.** *There exist integers  $m_0, \beta > 0$  depending on  $f$  such that the following holds. Let  $\mathcal{S}$  be a finite set of primes  $p > m_0$ , good and totally split in  $\widehat{k}_g/\mathbb{Q}$ , each given with positive integers  $d_{p,1} \dots, d_{p,s_p}$  (possibly repeated) such that  $d_{p,1}^1 \dots d_{p,s_p}^1$  is the type of some element in  $\nu(\overline{G})$ . Then there exists an integer  $b \in \mathbb{Z}$  such that*

- (i)  $0 \leq b \leq \beta \prod_{p \in \mathcal{S}} p$ ,
- (ii) *for each integer  $t_0 \equiv b \pmod{\beta \prod_{p \in \mathcal{S}} p}$ ,  $t_0$  is not a branch point of  $f$  and the collection of specializations of  $f$  at  $t_0$  is a single degree  $n$  field extension with residue degrees  $d_{p,1} \dots, d_{p,s_p}$  at  $p$  for each  $p \in \mathcal{S}$ .*

*Addendum 4.5* (on the constants) Denote the number of non-trivial conjugacy classes of  $\overline{G}$  by  $\text{cc}(\overline{G})$ . One can take  $m_0$  such that the interval  $[4r^2(n!)^2, m_0]$  contains at least  $\text{cc}(\overline{G})$  primes, good and totally split in  $\widehat{k}_g/\mathbb{Q}$ , and  $\beta$  to be the product of  $\text{cc}(\overline{G})$  such primes.

*Proof.* We use corollary 4.4 simultaneously for several primes: a first set of primes associated to all non-trivial elements of  $\overline{G}$  as explained above, and the set of primes given in the statement with the associated types. We apply the G-version of corollary 4.4 to the former data and the mere version to the latter. This provides an arithmetic progression  $(am + b)_{m \in \mathbb{Z}} \subset \mathbb{Z}$  with ratio  $a = \beta \prod_{p \in \mathcal{S}} p$  where  $\beta > 0$  is the product of the primes in the first set. The primes dividing  $\beta$  guarantee that the

collection of specializations at  $am + b$  of the  $k$ -cover  $f$  is a single field extension  $E/k$  of degree  $n$ . And each of the primes  $p \in \mathcal{S}$  gives that the  $\mathbb{Q}_p$ -étale algebra  $E \otimes_k \mathbb{Q}_p$  has degree divisor  $d_{p,1}^1 \cdots d_{p,s_p}^1$ .  $\square$

**4.4. Ample fields.** Recall that a field  $k$  is said to be *ample* if every smooth  $k$ -curve with a  $k$ -rational point has infinitely many  $k$ -rational points. Over an ample field, the twisting lemma 3.4 yields the following statement which generalizes §3.3.2 (\*\*\*) from [Dèb99].

**Corollary 4.6.** *Let  $k$  be an ample field and  $f : X \rightarrow B$  be a degree  $n$   $k$ -cover of curves. Let  $t_0 \in B(k)$  not in the branch point set  $\mathbf{t}$ . There exist infinitely many  $t \in B(k) \setminus \mathbf{t}$  such that the collection of specializations  $\prod_l k(X)_{t,l}/k$  and  $\prod_l k(X)_{t_0,l}/k$  at  $t$  and  $t_0$  respectively are equal.*

*Proof.* Take the  $k$ -étale algebra  $\prod_{l=1}^s E_l/k$  from lemma 3.4 to be the collection  $\prod_{l=1}^s k(X)_{t_0,l}/k$  of specializations at  $t_0$ . With the notation from §3.1, we have  $\varphi = \phi \circ \mathbf{s}_{t_0}$  and  $\overline{\varphi} = \Lambda$ . Hence condition (const/comp) holds with  $\overline{\chi} = \text{Id}_{G/\overline{G}}$ , and  $\Gamma \neq \emptyset$  (remark 3.3 (a)). From implication (i)  $\Rightarrow$  (ii) in lemma 3.4, there exists  $\gamma \in \Gamma$  such that conditions (ii-1) and (ii-2) are satisfied for  $t_0$  with  $\chi = \chi_\gamma$ . Condition (ii-1) is that there exists  $x_0 \in \widetilde{Z}^{\chi_\varphi}(k)$  with  $\widetilde{g}^{\chi_\varphi}(x_0) = t_0$ . As  $k$  is ample and  $\widetilde{Z}^{\chi_\varphi}$  is a smooth  $k$ -curve, there are infinitely many  $k$ -rational points  $x$  on  $\widetilde{Z}^{\chi_\varphi}$ . The corresponding points  $t = \widetilde{g}^{\chi_\varphi}(x) \in B(k)$ , excluding the branch points, satisfy conditions (ii-1) and (ii-2) from lemma 3.4. Implication (ii)  $\Rightarrow$  (i) of this lemma finishes the proof.  $\square$

*Remark 4.7.* The proof and the result generalize to higher dimensional covers  $f : X \rightarrow B$ . It should be assumed however that the covering space  $Z^{\text{sep}}$  of the cover  $Z^{\text{sep}} \rightarrow B \otimes_k k^{\text{sep}}$  corresponding to the field extension  $k^{\text{sep}}(Z)/k^{\text{sep}}(B)$  is smooth ( $Z^{\text{sep}}$  is the normalization of  $B$  in the field  $k^{\text{sep}}(Z)$  (defined in §2.2) and so is *a priori* only normal). The ampleness of  $k$  then provides a Zariski-dense subset of  $k$ -rational points on  $\widetilde{Z}^{\chi_\varphi}$  and the conclusion becomes that there exists a Zariski-dense subset  $\mathcal{B} \subset B(k) \setminus D$  such that the collection of specializations  $\prod_l k(X)_{t,l}/k$  at each  $t \in \mathcal{B}$  equals  $\prod_l k(X)_{t_0,l}/k$ .

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